

Rational R -matrices, centralizer algebras and tensor identities for e_6 and e_7 exceptional families of Lie algebras

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Abstract

We use Cvitanović's diagrammatic techniques to construct the rational solutions of the Yang-Baxter equation associated with the e_6 and e_7 families of Lie algebras, and thus explain Westbury's observations about their uniform spectral decompositions. In doing so we explore the extensions of the Brauer and symmetric group algebras to the centralizer algebras of e_7 and e_6 on their lowest-dimensional representations and (up to three-fold) tensor products thereof, giving bases for them and a range of identities satisfied by the algebras' defining invariant tensors.

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1 Introduction

The Yang-Baxter equation (YBE) [1], which appears in $1+1D$ physics as the factorizability condition for S -matrices in integrable models, is closely bound up with Lie algebras and their representation theory, essentially because of the asymptotic behaviour of its rational solutions (see (2.3) below). Indeed, if one were to investigate the YBE knowing nothing of Lie algebras, one would very soon find oneself re-discovering a great deal about them. In fact surprisingly little is known about the rational YBE solutions associated with the exceptional Lie algebras: this paper investigates these, and finds, in precisely the spirit of the preceding sentence, an intricate relationship between the YBE and various identities satisfied by the algebras' invariant tensors.

Our point of departure is a remarkable observation made a few years ago by Westbury [2]: that certain solutions of the YBE (' R -matrices') associated with the Lie algebras of the e_6 and e_7 series (the second and third rows of the Freudenthal-Tits 'magic square') have spectral decompositions which may be expressed simply and uniformly in terms of the dimension ($= 1, 2, 4$ or 8) of the underlying division algebra. In this paper we shall explicitly construct these R -matrices, prove that they solve the Yang-Baxter equation, and thus provide an explanation of Westbury's observation. More interesting, perhaps, is what we shall learn along the way. In particular we will need to understand the structure of, and provide a basis for, the centralizer of the Lie group action on tensor cubes of the defining representation. (These are the analogues for the exceptional series of the symmetric group algebra for $su(n)$ and of Brauer's algebra for the other classical groups.) We shall also discover a host of secondary identities satisfied by the groups' defining invariant tensors, all of them subtly necessary in solving the YBE.

Our method is to use Cvitanović's 'birdtrack' diagrams [3, 4] (which extend earlier ideas of Penrose) to handle the calculations. An alternative approach to the e_n centralizers, which utilizes the braid matrices (the q -deformed but spectral-parameter u -independent R -matrices) and is complementary to ours, appears in [5].

The paper is structured as follows. In section two we provide a brief recapitulation of some of Westbury's observations. In section three we give an elementary, essentially pedagogical recapitulation of these issues for the classical groups – the rational R -matrices, the centralizer algebras and the diagrammatic techniques used to handle them. Section four deals with the e_6 series, and section five with the e_7 series.

2 The Yang-Baxter equation and a unified spectral decomposition for exceptional R -matrices

2.1 The Yang-Baxter equation

The Yang-Baxter equation (YBE), between expressions in $\text{End}(V \otimes V \otimes V)$ for $V = \mathbb{C}^n$, is

$$\check{R}(u) \otimes 1 \cdot 1 \otimes \check{R}(u+v) \cdot \check{R}(v) \otimes 1 = 1 \otimes \check{R}(v) \cdot \check{R}(u+v) \otimes 1 \cdot 1 \otimes \check{R}(u), \quad (2.1)$$

or, with its indices made explicit (each running from 1 to n , and with repeated indices summed),

$$\check{R}_{lm}^{ij}(u) \check{R}_{sr}^{mk}(u+v) \check{R}_{pq}^{ls}(v) = \check{R}_{lm}^{jk}(v) \check{R}_{ps}^{il}(u+v) \check{R}_{qr}^{sm}(u), \quad (2.2)$$

for $\check{R}(u) \in \text{End}(V \otimes V)$. We first note that this equation is homogeneous in \check{R} and in u , so that $\mu \check{R}(\lambda u)$ is still a solution for arbitrary \mathbb{C} -scalings λ and μ . We shall therefore rescale both \check{R} and u wherever it is convenient for us to do so. (In the physical construction of factorized S -matrices, in contrast, the scale of u is fixed, and scaling \check{R} affects its analytic properties and thus the bootstrap spectrum.)

The simplest class of solutions of the YBE (which we refer to as ' R -matrices') has rational dependence on u , and an expansion in powers of $1/u$ of the form

$$\check{R}(u) = \mathbf{P} \left(1_n \otimes 1_n + \frac{C}{u} + \dots \right) \quad \text{where} \quad C = \sum_{a,b} \rho_V(I^a) \otimes \rho_V(I^b) g_{ab}, \quad (2.3)$$

in which 1_n is the $n \times n$ identity matrix, I^a are the generators of a Lie algebra \mathfrak{g} , g_{ab} its Cartan-Killing form, ρ_V its suitably-chosen representation on a module V (usually its

defining representation), and \mathbf{P} the transposition operator on the two components of $V \otimes V$. Thus, from the outset, the investigation of R -matrices naturally involves the investigation of Lie algebras and their representations.

A natural consequence of this (see, for example, [6]) is that $\check{R}(u)$ commutes with the action of \mathfrak{g} on $V \otimes V$, so that, by Schur's lemma,

$$\check{R}(u) = \sum_i f_i(u) P_i \quad (2.4)$$

for some scalar functions $f_i(u)$, where the sum is over projectors onto irreducible components $W_i \subset V \otimes V$. (This is only fully correct where there are no multiplicities. Where such repetitions among the W_i occur, there can be non-trivial intertwiners between them.)

2.2 R -matrix spectra and the magic square

Now recall that the Freudenthal-Tits ‘magic square’ [7, 8] is

$m =$	1	2	4	8
	a_1	a_2	c_3	f_4
	a_2	$a_2 \times a_2$	a_5	e_6
	c_3	a_5	d_6	e_7
	f_4	e_6	e_7	e_8

(We will not need the details of its construction. For full discussions, including an explanation of its row \leftrightarrow column symmetry, see [9, 10].) We will refer to the row whose last ($m = 8$) entry is the exceptional algebra \mathfrak{g} as the ‘ \mathfrak{g} series’ of Lie algebras.

For the e_6 series, Westbury's principal observation in [2] was that, in the literature of rational R -matrix spectral decompositions for individual \mathfrak{g} and V (originally in [11] for a_n , [12] for e_6), there is a unified underlying formula: for the representation on V of dimension $n = 3m + 3$,

$$\check{R}(u) = P_1 + \frac{4+u}{4-u} P_2 + \frac{4+u}{4-u} \frac{2m+u}{2m-u} P_3, \quad (2.5)$$

where W_1 is the representation whose highest weight is double that of V , W_2 is the anti-symmetric component of $V \otimes V$, and $W_3 = \bar{V}$, the complex-conjugate of V .

The YBE is straightforwardly generalized to act on $V_1 \otimes V_2 \otimes V_3$ for $V_1 \neq V_2 \neq V_3$. There is then a unified spectral decomposition for $\mathbf{P}\check{R}_{V\bar{V}}(u) \in \text{End}(V \otimes \bar{V})$ (in which \mathbf{P} now transposes elements of $V \otimes \bar{V}$ with those of $\bar{V} \otimes V$), for which

$$\check{R}_{V\bar{V}}(u) = \mathbf{P} \left(P_1 + \frac{u+m+4}{u-m-4} P_2 + \frac{u+m+4}{u-m-4} \frac{u+3m}{u-3m} P_3 \right), \quad (2.6)$$

where W_1 is the representation whose highest weight is the sum of those of V and \bar{V} , $W_2 = \mathfrak{g}$, the adjoint representation, and $W_3 = \mathbb{C}$, the singlet.

For the e_7 series, Westbury observes that, for V of dimension $n = 6m + 8$,

$$\check{R}(u) = P_1 + \frac{2+u}{2-u} P_2 + \frac{2+u}{2-u} \frac{m+2+u}{m+2-u} P_3 + \frac{2+u}{2-u} \frac{m+2+u}{m+2-u} \frac{2m+2+u}{2m+2-u} P_4, \quad (2.7)$$

where the highest weight of W_1 is twice that of V , W_2 is the highest antisymmetric component of $V \otimes V$, $W_3 = \mathfrak{g}$ and $W_4 = \mathbb{C}$. (The original R -matrix spectra are in [13] for c_3 , [11] for a_5 , [14] for d_6 and [12] for e_7 ; see also [16] for an extension to further values of m .)

We shall not, in this paper, concern ourselves with the g_2 series (the ‘zeroth’ row of the magic square, for which the R -matrices are dealt with in [15, 17]), or the f_4 and e_8 series, which are each, in different ways, problematic.

For the f_4 series, where the same observation might be expected to hold for V of dimension $3m + 2$, in fact (surprisingly) it fails. A uniform decomposition exists for c_3 and f_4 , but fails to work fully for the other algebras in the series. We suspect that the resolution is bound up with the identities satisfied by the primitive invariant tensor, and are working to understand this. A common feature of the f_4 and e_8 calculations is the need to evaluate ‘pentagon’ diagrams (in the diagrammatic notation of the later sections).

The e_8 series (which, suitably extended, includes all of the exceptional Lie algebras) is the most intriguing. For e_8 , the smallest representation on which an R -matrix may be constructed (and in fact the smallest representation of the Yangian $Y(e_8)$ [18]) is the \mathfrak{g} -reducible representation $\mathfrak{g} \oplus \mathbb{C}$. Its R -matrix is constructed in [19], and Westbury observes that this has a nice, uniform parametrization by Vogel’s plane [20]. (Note that

such uniformity suggests an extension of Deligne's conjecture [21], about the uniformity of decomposition of $\mathfrak{g}^{\otimes r}$, to Yangians.) Although both conventional [22] and diagrammatic [3] techniques for the adjoint representation of the e_8 series (the latter as advocated in [23, 24]) are well-developed, we have not yet been able to extend them to this reducible representation. Such a treatment of the R -matrix remains, however, highly desirable, as a step towards explaining the remarkable appearance of spectra associated with the algebras of the e_8 series in the q -state Potts model S -matrix [25, 26].

Westbury's observations also apply to trigonometric (q -dependent) R -matrices when q is not a root of unity. As far as we know, the centralizer algebras we study, which q -deform to the Iwahori-Hecke algebra for the $su(n)$ and the Birman-Wenzl-Murakami algebra for the other classical cases, have not been constructed for exceptional \mathfrak{g} other than g_2 [27].

3 The classical Lie algebras

Perhaps the two best-known, classic solutions of the YBE are those of Yang [28], acting on the n -dimensional module of $SU(n)$,

$$\check{R}_{cd}^{ab}(u) = 2\delta_c^a\delta_d^b - u\delta_d^a\delta_c^b = (2\mathbf{1}_n \otimes \mathbf{1}_n - u\mathbf{P})_{cd}^{ab}, \quad (3.1)$$

and of the Zamolodchikovs [29], acting on the n -dimensional module of $SO(n)$,

$$\check{R}_{cd}^{ab}(u) = 2\delta_c^a\delta_d^b - u\delta_d^a\delta_c^b + \frac{2u}{n-2-u}\delta_b^a\delta_d^c. \quad (3.2)$$

In the classic diagrammatic notation for these, which avoids a proliferation of indices in calculations, (3.1) is

$$\begin{array}{c} a \quad c \\ \diagdown \quad \diagup \\ \textcircled{u} \\ \diagup \quad \diagdown \\ b \quad d \end{array} = 2 \begin{array}{c} \text{---} \\ \text{---} \end{array} - u \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \quad (3.3)$$

and (3.2) is

$$\begin{array}{c} a \quad c \\ \diagdown \quad \diagup \\ \textcircled{u} \\ \diagup \quad \diagdown \\ b \quad d \end{array} = 2 \begin{array}{c} \text{---} \\ \text{---} \end{array} - u \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} + \frac{2u}{n-2-u} \bigg) \bigg(, \quad (3.4)$$

in which each Kronecker delta is written as a line connecting two indices. Concatenation of

symbols (by connecting lines, horizontally, from right to left) is the correct way to multiply these (since $\delta_b^a \delta_c^b = \delta_c^a$), so that the YBE becomes

$$\begin{array}{c} \text{---} \circ u \text{---} \\ \text{---} \circ u+v \text{---} \\ \text{---} \circ v \text{---} \end{array} = \begin{array}{c} \text{---} \circ u+v \text{---} \\ \text{---} \circ v \text{---} \\ \text{---} \circ u \text{---} \end{array}, \quad (3.5)$$

in which internal lines represent summed indices and external lines free indices. Checking that (3.1,3.2) are indeed solutions is now a matter of checking the equivalence of two \mathbb{C} -linear combinations of symbols, subject in the latter case to the further condition that a loop takes value $\delta_b^a \delta_a^b = n$.

3.1 $su(n)$

As already indicated, there is a Lie algebra and its representation theory underlying each of these solutions. In the first case, and denoting the n -dimensional module of $SU(n)$ by V and its conjugate by \bar{V} , we re-write (3.1,3.3) as

$$\begin{array}{c} a \swarrow \\ \circ u \\ b \swarrow \end{array} \begin{array}{c} \nwarrow c \\ \nwarrow d \end{array} = (2-u)P_+ + (2+u)P_- \quad (3.6)$$

where

$$P_{\pm} = \frac{1}{2} \left(\begin{array}{c} \leftarrow \\ \leftarrow \end{array} \pm \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) \quad (3.7)$$

are idempotents $P_{\pm}^2 = P_{\pm}$ (and we henceforth distinguish V from \bar{V} by decorating each line with an arrow). In fact these are the projectors onto the symmetric and antisymmetric irreducible components of the tensor square $V \otimes V$, and we thus have the spectral decomposition of the R -matrix, in form (2.4).

The R -matrix takes its values in the centralizer algebra $End_G(V^{\otimes 2})$, the commutant of the action of the group G (and Lie algebra \mathfrak{g}) on $V \otimes V$. The projectors (3.7), therefore, or alternatively $\delta_c^a \delta_d^b$ and $\delta_d^a \delta_c^b$ and the symbols which represent them, form a basis for $End_{su(n)}(V^{\otimes 2}) = \mathbb{C}S_2$, the algebra of the symmetric group S_2 . Similarly the YBE is an

equation of expressions in $End_{su(n)}(V^{\otimes 3}) = \mathbb{C}S_3$, or, in the symbolic notation, \mathbb{C} -linear sums of the symbols

$$\begin{array}{c} \overleftarrow{\quad} \\ \overleftarrow{\quad} \\ \overleftarrow{\quad} \end{array}, \quad \begin{array}{c} \diagup \\ \diagdown \end{array}, \quad \begin{array}{c} \overleftarrow{\quad} \\ \diagdown \end{array}, \quad \begin{array}{c} \diagup \\ \diagdown \end{array}, \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \text{ and } \begin{array}{c} \diagup \\ \diagdown \end{array}. \quad (3.8)$$

This mutually-centralizing action of S_p and $SU(n)$ on $V^{\otimes p}$ is the classic Schur-Weyl duality.

3.2 $so(n)$

We can rewrite (3.2,3.4) similarly as

$$\begin{array}{c} a \\ \diagdown \\ \textcircled{u} \\ \diagup \\ b \end{array} \begin{array}{c} c \\ \diagup \\ \diagdown \\ d \end{array} = (2-u)P_+ + (2+u)P_- + (2+u)\frac{n-2+u}{n-2-u}P_0, \quad (3.9)$$

where

$$P_+ = \frac{1}{2} \left(\begin{array}{c} \overline{\quad} \\ \overline{\quad} \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} \right) - \frac{1}{n} \bigg) \bigg(, \quad P_- = \frac{1}{2} \left(\begin{array}{c} \overline{\quad} \\ \overline{\quad} \end{array} - \begin{array}{c} \diagup \\ \diagdown \end{array} \right), \quad P_0 = \frac{1}{n} \bigg) \bigg(\quad (3.10)$$

are the projectors onto the symmetric traceless, antisymmetric and singlet components of the tensor square $V \otimes V$ of the defining, n -dimensional representation of $so(n)$. To check that each $P^2 = P$, we need the algebraic relations among these symbols, which are simply those of concatenation together with the loop value n , or

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} = \overline{\quad}, \quad \overline{\quad} \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \overline{\quad} = \begin{array}{c} \diagup \\ \diagdown \end{array},$$

$$\overline{\quad} \bigg(= \bigg) \overline{\quad} = \begin{array}{c} \diagup \\ \diagdown \end{array} \bigg(= \bigg) \overline{\quad} = \bigg) \bigg(, \quad \bigg) \bigcirc \bigg(= n \bigg) \bigg(.$$

The dimension of the module corresponding to the idempotent P is computed in the algebra by connecting the in- to the out- top index and the in- to the out- bottom index, equivalent

to taking the trace in the tensor product by setting $a = c$ and $b = d$ and summing. This gives values for P_+ , P_- and P_0 of $n(n+1)/2 - 1$, $n(n-1)/2$ and 1 respectively.

This algebra, $End_{so(n)}(V^{\otimes 2})$, is Brauer's algebra $B_2(n)$ [30, 31]. The YBE is now valued in $End_{so(n)}(V^{\otimes 3}) = B_3(n)$, the 15-dimensional algebra spanned by

subject to the same rules of concatenation and loop value n .

3.3 $sp(2r)$

There is another solution of the YBE [32], associated to $sp(2r)$, which utilizes $B_2(-2r)$, although we shall instead write it in a form which makes the role of the symplectic form matrix explicit. It is

$$\begin{aligned}
 \begin{array}{c} a \\ \diagup \\ \textcircled{u} \\ \diagdown \\ b \end{array} &= (2-u)P_+ + (2+u)P_- + (2-u)\frac{2r+2+u}{2r+2-u}P_0 \\
 &= 2 \begin{array}{c} \text{---} \\ \text{---} \end{array} - u \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \frac{2u}{2r+2-u} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}, \quad (3.11)
 \end{aligned}$$

where

$$P_+ = \frac{1}{2} \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right), \quad P_- = \frac{1}{2} \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) + \frac{1}{2r} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}, \quad P_0 = -\frac{1}{2r} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}, \quad (3.12)$$

are the projectors onto the symmetric, antisymmetric and symplectic-traceless, and singlet components of $V \otimes V$. We use a solid arrow \curvearrowleft to denote the symplectic form matrix,

so that $\overleftarrow{\leftarrow\leftarrow\leftarrow} = -\text{---}$, $\overleftarrow{\leftarrow} = -\overrightarrow{\rightarrow}$ and $\overleftarrow{\leftarrow\rightarrow} = \text{---}$. If we denote an element of $Sp(2r)$ by $\overleftarrow{\bigcirc}$ then the defining relation $MJM^T = J$ for $M \in Sp(2r)$ is that

$$\overleftarrow{\bigcirc}\overleftarrow{\leftarrow\leftarrow\leftarrow}\overleftarrow{\bigcirc} = \overleftarrow{\leftarrow}. \quad (3.13)$$

The algebra $End_{sp(2r)}(V^{\otimes 2})$ is generated by the three symbols in (3.11), with the invariance of the third being due to (3.13),

$$\overleftarrow{\bigcirc}\overleftarrow{\leftarrow\leftarrow\leftarrow}\overleftarrow{\bigcirc} = \overleftarrow{\leftarrow}\overleftarrow{\leftarrow\leftarrow\leftarrow} = \overleftarrow{\leftarrow\leftarrow\leftarrow}\overleftarrow{\bigcirc}. \quad (3.14)$$

It is simple to check that each of the three given projectors is indeed idempotent. The YBE is valued in $End_{sp(2r)}(V^{\otimes 3})$, spanned by

$$\begin{array}{cccccc} \text{---} & \times & \text{---} & \times & \times & \times \\ \text{---} & \text{---} & \times & \times & \times & \times \\ \text{---} & \times & \times & \times & \times & \times \end{array}$$

3.4 Dimension of $End_{\mathfrak{g}}(V^{\otimes p})$

An alternative basis for the centralizer algebra $End_{\mathfrak{g}}(V^{\otimes p})$ (for semisimple \mathfrak{g}) is given by the set of projectors and intertwiners of \mathfrak{g} -irreducible components of $V^{\otimes p} = \bigoplus_i \mathbb{C}^{d_i} \otimes W_i$ (in which d_i is the multiplicity of W_i in the decomposition). Thus

$$\dim End_{\mathfrak{g}}(V^{\otimes p}) = \sum_i d_i^2, \quad (3.15)$$

which we shall find useful in dealing with the exceptional algebras, where a diagrammatic basis for $End_{\mathfrak{g}}(V^{\otimes 3})$ (as used in the previous subsections) will be far from obvious. The central utility of such bases, which is not achieved by using projectors and intertwiners, is to facilitate calculations in $End_{\mathfrak{g}}(V^{\otimes 3})$ using terms from the different embeddings of $End_{\mathfrak{g}}(V^{\otimes 2})$, as required by the YBE.

4 The e_6 series

The defining property of \mathfrak{g} in the e_6 series, as subgroups $G \subset SU(n)$ with $n = 3m + 3$, is the existence of a cubic, symmetric invariant form d_{abc} , *i.e.* a map $V^{\otimes 3} \rightarrow \mathbb{C}$, $(u^a, v^b, w^c) \mapsto d_{abc} u^a v^b w^c$ such that, for $M \in G$, $d_{def} M^{da} M^{eb} M^{fc} = d_{abc}$ or, in diagrammatic notation (and with $\text{---}\bigcirc\text{---}$ denoting M),

$$\begin{array}{c} \text{---}\bigcirc\text{---} \\ \text{---}\bigcirc\text{---} \\ \text{---}\bigcirc\text{---} \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \quad \text{and} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array}. \quad (4.1)$$

(This should not be confused with the cubic Casimir operator of $su(n)$ corresponding to the cubic symmetric invariant in the *adjoint* representation.) Following [3], this is normalized so that

$$\begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array}. \quad (4.2)$$

Thus the symmetric component of $V \otimes V$ decomposes further, and the R -matrix (2.5), with

$$P_1 = \frac{1}{2} \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} \right) - \begin{array}{c} \diagup \\ \diagdown \end{array}, \quad P_2 = \frac{1}{2} \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \diagup \\ \diagdown \end{array} \right), \quad P_3 = \begin{array}{c} \diagup \\ \diagdown \end{array}, \quad (4.3)$$

may be multiplied by $4 - u$ to give

$$\begin{array}{c} a \\ \swarrow \\ \bigcirc \\ \nwarrow \\ b \end{array} \begin{array}{c} c \\ \swarrow \\ \bigcirc \\ \nwarrow \\ d \end{array} = 4 \begin{array}{c} \text{---} \\ \text{---} \end{array} - u \begin{array}{c} \diagup \\ \diagdown \end{array} + \frac{(4m+8)u}{2m-u} \begin{array}{c} \diagup \\ \diagdown \end{array}. \quad (4.4)$$

(Note the re-scalings of R and u relative to the $su(n)$ R -matrix (3.1,3.3).)

As discussed in section 3.2, one computes the putative trace of an idempotent by connecting its in- and out- legs. For the idempotents constructed using d_{abc} this is an integer, and thus the centralizer algebra has an action on a module, when $n = 3m + 3$ for $m = 1, 2, 4$ and 8 (although not only for these—for the full story see [3]). It is worth noting that the centralizer algebras for all, including classical, \mathfrak{g} are formally defined, and R -matrices in them exist, for *all* n , not just integers: it is only the requirement that idempotents have integer ‘trace’ which further restricts n .

The three-dimensional centralizer $End_{\mathfrak{g}}(V \otimes V)$ is generated by the three symbols which appear in (4.4): note that the third symbol's commuting with E_6 follows from (4.1),

$$\begin{array}{c} \diagup \quad \diagdown \\ \hline \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \hline \diagup \quad \diagdown \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \hline \diagdown \quad \diagup \end{array}, \quad (4.5)$$

in which the bar denotes complex conjugation, so that the defining property of $U(n)$ is

$$\begin{array}{c} \leftarrow \quad \rightarrow \\ \hline \leftarrow \quad \rightarrow \end{array} = \leftarrow. \quad (4.6)$$

Our object now is to demonstrate that (2.5) satisfies the YBE. It is clear that this will include terms of even orders in d up to six, and (for reasons which will become apparent below) that there will be reduction relations among them. This will all be rather involved, and so we move now to a more mathematically-formal layout.

There are two primary identities satisfied by the invariant d_{abc} , at fourth and third order respectively, and there are no more at these or lower orders [3]. The first is

Lemma 4.1 (Cvitanović):

$$\begin{array}{c} \diagup \quad \diagdown \\ \hline \diagdown \quad \diagup \end{array} = -\frac{m}{2m+4} \begin{array}{c} \diagup \quad \diagdown \\ \hline \diagup \quad \diagdown \end{array} + \frac{1}{2m+4} \left(\begin{array}{c} \leftarrow \\ \hline \rightarrow \end{array} + \begin{array}{c} \leftarrow \\ \hline \leftarrow \end{array} \begin{array}{c} \rightarrow \\ \hline \rightarrow \end{array} \right).$$

Proof in [3], eqn. (18.9); follows from irreducibility of components of $V \otimes \bar{V}$. \square

All terms in the YBE are of rank six (where the rank, the number of free indices, is the number of external legs of a diagram), and to reduce the sixth-order terms in the YBE we will need

Corollary 4.2:

$$\begin{array}{c} \diagup \quad \diagdown \\ \hline \diagdown \quad \diagup \end{array} - \begin{array}{c} \leftarrow \quad \rightarrow \\ \hline \leftarrow \quad \rightarrow \end{array} = \frac{m}{2m+4} \left(\begin{array}{c} \leftarrow \quad \rightarrow \\ \hline \leftarrow \quad \rightarrow \end{array} - \begin{array}{c} \leftarrow \quad \rightarrow \\ \hline \leftarrow \quad \rightarrow \end{array} \right) - \frac{1}{2m+4} \left(\begin{array}{c} \leftarrow \quad \rightarrow \\ \hline \leftarrow \quad \rightarrow \end{array} - \begin{array}{c} \leftarrow \quad \rightarrow \\ \hline \leftarrow \quad \rightarrow \end{array} \right).$$

Proof is by applying Lemma 4.1 to the loops. \square

With the sixth-order terms thus reduced, we now deal with the fourth-order terms. To do so we begin with the other primary identity satisfied by the invariant d ,

Lemma 4.3 (Freudenthal):

$$\left\{ \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \end{array} \right\} \text{symmetrized} = \frac{4}{3m+6} \left\{ \begin{array}{c} \text{diagram 4} \\ \text{diagram 5} \\ \text{diagram 6} \end{array} \right\} \text{symmetrized}.$$

Proof in [33], eqn. (1.17). The cubic invariant is the determinant of a 3×3 hermitian matrix X with entries in the division algebra of order m (and which thus form V of dimension $3m+3$). Freudenthal utilizes d to define a product $\times : V \otimes V \rightarrow \bar{V}$ (and its conjugate) which obeys $(X \times X) \times (X \times X) = X \det X$, expressed diagrammatically above. The relation appears in [3] (sect.18.10) as the ‘Springer relation’ [34], and for e_6 specifically in [4], Fig.15(b). \square

Once again we need relations of rank six rather than the rank-five of Lemma 4.3, and so must construct the secondary identity

Corollary 4.4:

$$\text{diagram 1} + \text{diagram 2} + \text{diagram 3} = \frac{1}{m+2} \left(\text{diagram 4} + \text{diagram 5} + \text{diagram 6} + \text{diagram 7} \right).$$

Proof by appending another copy of d to the diagrams of Lemma 4.3, expanding the symmetrizers, and re-arranging. \square

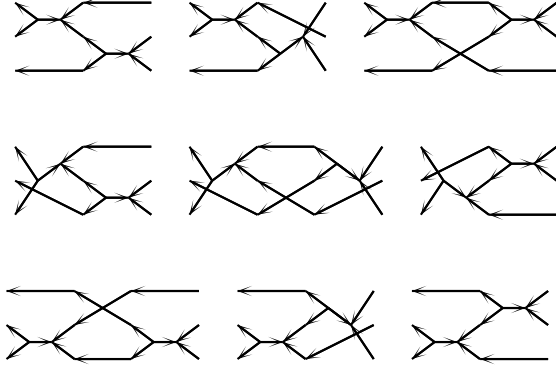
These results are sufficient for us now to prove

Theorem 4.5: $End_{e_6}(V^{\otimes 3})$ is the 20-dimensional algebra, with subalgebra $End_{su(27)}(V^{\otimes 3}) = \mathbb{C}S_3$, spanned by

$$\begin{array}{cccccc} \text{diagram 1} & \text{diagram 2} & \text{diagram 3} & \text{diagram 4} & \text{diagram 5} & \text{diagram 6} \end{array},$$



and




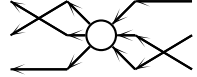

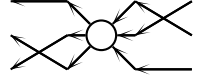
Proof. The six symbols at zeroth order and the ten at second order are trivially independent – they cannot be related by Lemma 4.3. At fourth order let us denote the nine symbols by e_{ij} , $i, j = 1, 2, 3$, where i (respectively j) indexes the left-hand, V (resp. right-hand, \bar{V}) leg not contracted on a common d . Corollary 4.4 (for $m = 8$) then reduces $e_{11} + e_{12} + e_{13}$ to terms of lower order. Permuting external legs (and \mathbb{C} -conjugating where necessary) gives six such reduction relations in total, reducing $e_{i1} + e_{i2} + e_{i3}$ and $e_{1i} + e_{2i} + e_{3i}$ for each $i = 1, 2, 3$. Only five of these six are independent, since $\sum_i (e_{i1} + e_{i2} + e_{i3} - e_{1i} - e_{2i} - e_{3i}) = 0$. There are therefore five reduction relations among the nine symbols at fourth order, leaving four independent generators. We thus have $\dim \text{End}_{\mathfrak{g}}(V^{\otimes 3}) = 6 + 10 + 4 = 20$, matching that computed from (3.15). \square

Remark 4.6. A set of four independent symbols among the nine at fourth order is furnished by any set of four which neither (i) includes three from any single row or column, nor (ii) consists of two from one row and the other two from the excluded column. An example is $\{e_{11}, e_{12}, e_{21}, e_{22}\}$.

Remark 4.7. Theorem 4.5 does not apply to other \mathfrak{g} in the e_6 series, for which there is a further reduction (which does not affect our YBE results). For details, and an extended Young tableau method for the e_6 series, see ch.18 of [3].

For the YBE we will need some further fourth-order relations, for which we begin with

Definition 4.8: for any rank-six symbol  we define the transformations

$$T_1 : \text{  } \mapsto \text{  }, \quad T_2 : \text{  } \mapsto \text{  },$$

which facilitates










Lemma 4.9: the unique (up to scaling) fourth-order term of rank six with eigenvalue -1 under both T_1 and T_2 is

$$\text{  } := \text{  } - \text{  } + \text{  } - \text{  } + \text{  } - \text{  }.$$

Proof by direct calculation. □

Next is the key lemma in checking the YBE,

Lemma 4.10:

$$\begin{aligned} \text{  } - \text{  } &= \frac{1}{3} \text{  } + \frac{2}{3m+6} \left(\text{  } - \text{  } \right) \\ &+ \frac{1}{3m+6} \left(\text{  } + \text{  } - \text{  } - \text{  } \right). \end{aligned}$$

Proof. This is a linear combination of four of the six variants of Corollary 4.4. Using again the basis e_{11}, \dots, e_{33} introduced in Theorem 4.5 for the fourth-order terms, it is the reduction formula for $\frac{1}{3} \sum_i (e_{1i} + e_{i1} - e_{3i} - e_{i3})$. □


It is such combinations of diagrams, and permutations (of the external legs) thereof, which appear in the Yang-Baxter equation, which we can now see is connected rather subtly, through the secondary identities in Corollary 4.4 and Lemma 4.10, with Freudenthal's primary relation, Lemma 4.3. Thus we can now prove

Theorem 4.11: the R -matrix (4.4) solves the YBE.

Proof. We first substitute (4.4) into the YBE (3.5) and expand the left-hand- minus the right-hand-side. The combination of sixth-order terms is precisely the left-hand-side of

Corollary 4.2, which thereby reduces the overall expression to fourth-order. There are then three combinations of fourth-order terms which appear,

$$\begin{array}{c} \text{Diagram 1} - \text{Diagram 2}, \quad \text{Diagram 3} - \text{Diagram 4} \quad \text{and} \quad \text{Diagram 5} - \text{Diagram 6}. \end{array}$$

To the first of these we apply Lemma 4.10, and to the others, respectively, T_1 and T_2 of Lemma 4.10. The nice behaviour of  under T_1 and T_2 ensures that its coefficient vanishes. What remains is a linear combination of the zeroth- and second-order symbols. That each of the coefficients vanishes was checked both by hand and using Maple. \square

Corollary 4.12: the $V \otimes \bar{V}$ R -matrix (2.6), with projectors

$$\begin{aligned} \mathbf{P}P_1 &= \frac{m+2}{m+4} \left(\text{Diagram 1} - \frac{1}{m+1} \text{Diagram 2} \right) \left(+ 2 \text{Diagram 3} \right), \\ \mathbf{P}P_2 &= \frac{2}{m+4} \left(\text{Diagram 1} + \frac{1}{3} \text{Diagram 2} \right) \left(- (m+2) \text{Diagram 3} \right) \quad \text{and} \quad \mathbf{P}P_3 = \frac{1}{3m+3} \left(\text{Diagram 1} \right) \left(\text{Diagram 2} \right) \end{aligned}$$

(from [3]) and thus (rescaled)

$$\begin{array}{c} a \\ \nearrow \\ \text{Diagram 1} \\ \nwarrow \\ b \end{array} \begin{array}{c} c \\ \nwarrow \\ \text{Diagram 1} \\ \nearrow \\ d \end{array} = \left(\text{Diagram 1} - 4 \frac{m+2}{u-m} \text{Diagram 2} + \frac{4}{u-3m} \text{Diagram 3} \right) \left(\text{Diagram 4} \right),$$

combines with the $V \otimes V$ R -matrix of Theorem 4.10 to solve the YBE on $V \otimes V \otimes \bar{V}$.

Proof. We rely here on the crossing-relation from factorized S -matrix theory (see, for example, [35]), which in our case states that

$$R_{V\bar{V}}(u) \propto \text{Cross}(R_{VV}(3m-u)),$$

where the operation Cross simply rotates the symbolic representation of R anticlockwise through 90° . It is simple to check that this holds, thereby implying that (2.6) is indeed the correct R -matrix on $V \otimes \bar{V}$. \square

5 The e_7 series

The progression of ideas in this section is very similar to that in the last. We begin by recalling that the defining property of the e_7 series, realized as subgroups $G \subset Sp(2r)$, is the existence of a symmetric, quartic invariant d_{pqrs} in the defining, n -dimensional module V (where $n = 2r = 6m + 8$). The invariance, in diagrammatic notation, is

$$\begin{array}{c} \circ \\ \diagup \\ \bullet \\ \diagdown \\ \circ \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array}. \quad (5.1)$$

(In contrast to the last section, we use a black disc to denote this quartic tensor, to avoid confusion with the transposition diagram.) The idempotents have integer trace here for $\dim V = n = 6m + 8$ ($m = 1, 2, 4, 8$).

Once again the symmetric component of $V \otimes V$ now decomposes further, modifying the projectors of section 3.3. The projectors in the R -matrix (2.7) (from [3], but here rendered symbolically) are

$$\begin{aligned} P_1 &= \frac{1}{6(m+4)} \left\{ 3(m+3) \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} \right) - \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \right\}, \\ P_2 &= \frac{1}{2} \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \diagup \\ \diagdown \end{array} \right) + \frac{1}{6m+8} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}, \\ P_3 &= \frac{1}{6(m+4)} \left\{ 3 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} \right) + \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \right\}, \quad P_4 = -\frac{1}{6m+8} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}. \end{aligned}$$

The R -matrix (2.7) is then, after re-scaling,

$$\begin{array}{c} a \\ \diagup \\ \circ \\ \diagdown \\ b \end{array} \begin{array}{c} c \\ \diagup \\ \circ \\ \diagdown \\ d \end{array} = (2m+4-u) \begin{array}{c} \text{---} \\ \text{---} \end{array} + u(u-m-1) \begin{array}{c} \diagup \\ \diagdown \end{array} + \frac{u(2+u)}{2m+2-u} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} + \frac{u}{3} \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array},$$

and we see that the four-dimensional $\text{End}_{\mathfrak{g}}(V \otimes V)$ for the e_7 series is generated by these four symbols. The only subtlety is in combining the symplectic form with d in the last

symbol: this is done so that

$$\begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array},$$

in which we have used both (5.1) and (3.13).

Our object is to demonstrate that (2.7) satisfies the YBE, and a similar story of tensor identities to that for the e_6 series now follows. As before, there are two primary identities, this time both of second order.

Lemma 5.1 (Cvitanović):

$$\begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} = 6(m+2) \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} + 18(m+3) \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \times \end{array} \right).$$

Proof: in [3], ch. 20, and specifically for e_7 in [4], Fig.18(e). □

Note that this, together with the relations of section 3.3 and

$$\begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} = 0, \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} = \begin{array}{c} \times \end{array},$$

fixes the structure of $\text{End}_{\mathfrak{g}}(V^{\otimes 2})$.

Lemma 5.2 (Brown):

$$\begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} + \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} = \\ 3 \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} + \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} + \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} + \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} - \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \right)$$

Proof: [36], section 3. Analogously to Freudenthal's relation for the e_6 series, Brown uses the quartic invariant to define an invariant map $V^{\otimes 3} \rightarrow V$, of which this is the key property. The primitive quartic invariant naturally occurs as the contraction of the symplectic (2-)form with the alternating (6-)form. The relation appears diagrammatically in [3] and [4], Fig.15(d). □

These identities are sufficient to prove

Theorem 5.3: $End_{e_7}(V^{\otimes 3})$ is the 35-dimensional algebra, with Brauer subalgebra $End_{sp(56)}(V^{\otimes 3})$, spanned by

and

Proof. The fifteen symbols of zeroth order and the fifteen of first order are independent; they cannot be related by Lemmas 5.1, 5.2. The terms at second order are, however, subject to Lemma 5.2 and its variants obtained by permuting external legs. Simple combinatorics superficially yields twenty-four of these, six rotations multiplied by the four possibilities of transposing (or not) the upper-left and upper-right pairs of legs (the only pairs not already related by symmetry in Lemma 5.2). However, Maple informs us that only five of the 24 variants are independent. Thus the ten symbols at second order are reduced by five independent reduction relations to five independent symbols, and we have $\dim End_{\mathfrak{g}}(V^{\otimes 3}) = 15 + 15 + 5 = 35$, matching the computation (3.15). \square

Remark 5.4. In contrast to the analogous result for e_6 (Remark 4.6), we do not here have a neat general characterization of all possible choices for five independent terms among

the ten at second order. However, from the form of Lemma 5.2 and its variants it is straightforward to argue that either (i) any one of the six first-row symbols together with the four others, or (ii) any four of the six first-row symbols together with any one of the next three, is likely to furnish an independent set. That this is indeed so was checked, for all such choices, using Maple.

Before proving the key reduction relations for the YBE, we first note

Lemma 5.5:

$$\text{Diagram 1} := \text{Diagram 2} + \text{Diagram 3} - \text{Diagram 4} - \text{Diagram 5} - \text{Diagram 6} + \text{Diagram 7}$$

is the unique second-order, rank-six term which is invariant under 60° rotations.

Proof by direct calculation. □

The relations essential for the YBE are then

Lemma 5.6


$$\begin{aligned} \text{Diagram 8} - \text{Diagram 9} &= \frac{1}{3} \text{Diagram 1} + 2 \left(\text{Diagram 10} - \text{Diagram 11} - \text{Diagram 12} - \text{Diagram 13} \right) \\ &+ \text{Diagram 14} + \text{Diagram 15} - \text{Diagram 16} - \text{Diagram 17} - \text{Diagram 18} - \text{Diagram 19} - \text{Diagram 20} - \text{Diagram 21} \end{aligned}$$

Proof. Write R_θ for the anticlockwise rotation of a symbol by angle θ . Then this is $(R_{60^\circ} + R_{120^\circ} - R_{240^\circ} - R_{300^\circ})$ of Lemma 5.2. □

Lemma 5.7


$$\begin{aligned} \text{Diagram 22} - \text{Diagram 23} &= 27(m+3) \left(\text{Diagram 24} - \text{Diagram 25} + \text{Diagram 26} - \text{Diagram 27} + \text{Diagram 28} + \text{Diagram 29} - \text{Diagram 30} - \text{Diagram 31} \right) \\ &- 3(2m+5) \left\{ 2 \left(\text{Diagram 32} + \text{Diagram 33} + \text{Diagram 34} - \text{Diagram 35} + \text{Diagram 36} - \text{Diagram 37} \right) \right. \end{aligned}$$

$$\left\{ \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \\ \text{diagram 4} \\ \text{diagram 5} \\ \text{diagram 6} \end{array} \right\} - (m+1) \text{diagram 7}$$


Proof. First we contract Lemma 5.6, on its top two indices, with the bottom two indices of , and use Lemma 5.1. This gives the antisymmetrization of Lemma 5.7 on its bottom two indices. Requiring 60° rotational symmetry then forces the required result. \square

With these secondary identities of the invariant tensor established, we can now prove

Theorem 5.8: the R -matrix (2.7) solves the YBE.

Proof. Again we first expand the left-hand- minus right-hand-side of the YBE (3.5) with (2.7) substituted, and then multiply by . The third-order terms are reduced by Lemma 5.7. Some of the second-order terms are reduced by Lemma 5.1 alone; the others are the differences

$$\begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \end{array} - \begin{array}{c} \text{diagram 3} \\ \text{diagram 4} \end{array}, \quad \begin{array}{c} \text{diagram 5} \\ \text{diagram 6} \end{array} - \begin{array}{c} \text{diagram 7} \\ \text{diagram 8} \end{array} \quad \text{and} \quad \begin{array}{c} \text{diagram 9} \\ \text{diagram 10} \end{array} - \begin{array}{c} \text{diagram 11} \\ \text{diagram 12} \end{array},$$

or the left-hand side of Lemma 5.6 and its rotations by $\pm 120^\circ$. On using these the  terms vanish because of their invariant behaviour under rotations, Lemma 5.5. What remains is an expression in the zeroth- and first-order, 30-dimensional subalgebra of the centralizer. That each of the coefficients vanishes was checked using Maple. \square

6 Concluding remarks

In constructing and verifying the rational R -matrices for the e_6 and e_7 series of Lie algebras, we have had to construct their centralizers on $V^{\otimes 3}$ as diagram algebras, and establish explicit bases for them (Theorem 4.5 for e_6 and Theorem 5.3 for e_7). The connection between the algebras' defining invariant tensors (and the primary reduction relations satisfied by these) and the Yang-Baxter equation only appears through a number of elegantly symmetric secondary identities (Corollary 4.2 and Lemmas 4.4, 4.10 for e_6 , and Lemmas 5.6, 5.7 for e_7).

As we mentioned earlier, our primary goal for future work remains to understand the e_8 case and its possible connections with the q -state Potts model [25].

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